International Journal of Theoretical Physics, Vol. 32, No. 11, 1993

Evolution of a Velocity-Dependent Quantum Forced Anharmonic Oscillator

R. D. Khan,[1] Zhang Jialun,[1] Ding Sheng,[1] and Shen Wenda[1]

Received July 10, 1992

The exact solution to a velocity-dependent quantum forced anharmonic oscillator is derived by using integral operators and an iteration method. The study is carried out in operational form by use of the creation and annihilation operators of the oscillator. The time development of the displacement and momentum operators of the anharmonic oscillator is given. These operators are presented as a Laplace transform and a subsequent inverse Laplace transform of suitable functionals.

1. INTRODUCTION

Anharmonic models play a fundamental role in quantum optics and many other branches of physics. A great many papers on approximate solutions of the Schrödinger equation for the anharmonic oscillator are available in the literature. Energy levels and eigenfunctions are calculated by using the Rayleigh–Schrödinger perturbation method, the Krylov–Bogoliubov method of averaging, or thermodynamic perturbation theory. Recently, Carusotto (1988) has studied the single quartic anharmonic oscillator and obtained its analytical solution for the first time by applying the integral operators and iteration method.

In our previous work (Zhang *et al.*, 1992) Carusotto's theory and method are generalized to treat the time evolution of a quantum forced anharmonic oscillator and its exact solution is obtained. In the present paper, the study is further generalized to a velocity-dependent quantum forced anharmonic oscillator characterized by the Hamiltonian

$$\hat{\mathbf{H}} = \frac{1}{2m}\hat{\mathbf{p}}^2 + \frac{1}{2}\omega^2 m\hat{\mathbf{q}}^2 + \frac{\varepsilon}{4}\hat{\mathbf{q}}^4 + f(t)\hat{\mathbf{q}} + \mathbf{g}(t)\hat{\mathbf{p}} \tag{1}$$

[1]Department of Physics, Shanghai University of Science and Technology, Shanghai 201800, China.

0020-7748/93/1100-2023$07.00/0

where $f(t)\hat{\mathbf{q}}$ is the driven term, $\mathbf{g}(t)\hat{\mathbf{p}}$ is the velocity-dependent term, and $f(t)$ and $\mathbf{g}(t)$ are real functions of time. We shall investigate the time evolution of the velocity-dependent anharmonic oscillator and give the detailed treatment used to analyze the nonlinear problems. First we look for the solution of the equation of motion for the displacement operator of the oscillator. Then we establish a nonlinear first-order differential equation suitably correlated with the equation of motion. A comparison between the solution of this differential equation and that of the motion equation permits us to condense the resultant power series into an integral of an analytical function. After further calculation, we obtain a final expression in which the displacement operator appears in the form of a Laplace transform and a subsequent inverse Laplace transform of suitable functionals.

2. MATHEMATICAL TREATMENT

If the characteristic parameters β, γ, and χ are introduced, the Hamiltonian (1) becomes

$$\hat{\mathbf{H}} = \beta\hat{\mathbf{p}}^2 + \gamma\hat{\mathbf{q}}^2 + \chi\hat{\mathbf{q}}^4 + f(t)\hat{\mathbf{q}} + \mathbf{g}(t)\hat{\mathbf{p}} \tag{2}$$

In the Heisenberg picture, the displacement and momentum operators $\hat{\mathbf{q}}(t)$ and $\hat{\mathbf{p}}(t)$ of the velocity-dependent anharmonic oscillator obey the equations

$$\frac{d\hat{\mathbf{q}}(t)}{dt} = \frac{1}{i\hbar}[\hat{\mathbf{q}}, \hat{\mathbf{H}}] = 2\beta\hat{\mathbf{p}} + \mathbf{g}(t) \tag{3a}$$

$$\frac{d\hat{\mathbf{p}}(t)}{dt} = -2\hat{\mathbf{q}}(2\chi\hat{\mathbf{q}}^2 + \gamma) - f(t) \tag{3b}$$

Consequently, the displacement operator $\hat{\mathbf{q}}(t)$ satisfies the equation of motion

$$\frac{d^2\hat{\mathbf{q}}}{dt^2} = b(\hat{\mathbf{q}}) - 2\beta f(t) + \frac{dg(t)}{dt} \tag{4}$$

with

$$b(\hat{\mathbf{q}}) = -4\beta\hat{\mathbf{q}}(t)[2\chi\hat{\mathbf{q}}^2(t) + \gamma]$$

We must solve the nonlinear second-order differential equation (4) so as to obtain the time evolution of the operator $\hat{\mathbf{q}}(t)$. If the iteration method is used, the solution of equation (4) may be written as

$$\hat{\mathbf{q}}(t) = \hat{\mathbf{X}}(t) + \hat{\mathbf{I}}^2(t)b\{\hat{\mathbf{X}}(t) + \hat{\mathbf{I}}^2(t)b\{\ldots\}\} \tag{5}$$

where

$$\hat{\mathbf{X}}(t) = \hat{\mathbf{q}}(0) + 2\beta\hat{\mathbf{I}}(t)\hat{\mathbf{p}}(0) + \hat{\mathbf{K}}(t)$$

$$\hat{\mathbf{I}}(t) = \int_0^t dt_1, \qquad \hat{\mathbf{I}}^2(t) = \int_0^t \int_0^{t_1} dt_2\, dt_1, \qquad \hat{\mathbf{I}}^n(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n$$

$$\hat{\mathbf{K}}(t) = \hat{\mathbf{F}}(t) + \hat{\mathbf{G}}(t)$$

$$\hat{\mathbf{F}}(t) = -2\beta\hat{\mathbf{I}}^2(t)f(t) = -2\beta \sum_{n=0}^{\infty} f^n(0)\hat{\mathbf{I}}^{n+2}(t)$$

$$\hat{\mathbf{G}}(t) = \hat{\mathbf{I}}(t)g(t) = \sum_{n=0}^{\infty} g^n(0)\hat{\mathbf{I}}^{n+1}(t)$$

$$f^n(0) = \left.\frac{\partial^n f}{\partial t^n}\right|_{t=0}, \qquad g^n(0) = \left.\frac{\partial^n g}{\partial t^n}\right|_{t=0}$$

We have to find the analytical function to which the series (5) converges. To this end, we consider the series

$$\mathbf{B}_1(\tau; \xi) = \xi + \hat{\mathbf{I}}(\tau)b\{\xi + \hat{\mathbf{I}}(\tau)b\{\ldots\}\} \tag{6}$$

where ξ and τ are independent c-number variables. The function $\mathbf{B}_1(\tau; \xi)$ obeys the equation

$$\frac{\partial \mathbf{B}_1}{\partial \tau} = b\{\xi + \hat{\mathbf{I}}(\tau)b\{\ldots\}\} = b(\mathbf{B}_1) \tag{7}$$

with the initial condition

$$\mathbf{B}_1(\tau = 0; \xi) = \xi$$

Clearly the function $\mathbf{B}_1(\tau; \xi)$ can be written as

$$\mathbf{B}_1(\tau; \xi) = \exp[\tau b(\xi)\, d/d\xi]\, \xi \tag{8}$$

In fact, if we differentiate equation (8) with respect to τ, we have

$$\frac{\partial \mathbf{B}_1}{\partial \tau} = \exp\left[\tau b(\xi)\frac{d}{d\xi}\right] b(\xi) \tag{9}$$

Since equation (9) can be written as

$$\frac{\partial \mathbf{B}_1}{\partial \tau} = \exp\left[\tau b(\xi)\frac{d}{d\xi}\right] b(\xi) \cdot \exp\left[-\tau b(\xi)\frac{d}{d\xi}\right] \cdot \mathbf{1}$$

the function $\mathbf{B}_1(\tau; \xi)$ in equation (8) satisfies equation (7), i.e.,

$$\frac{\partial \mathbf{B}_1}{\partial \tau} = b\left\{\exp\left[\tau b(\xi)\frac{d}{d\xi}\right]\xi\right\} = b(\mathbf{B}_1)$$

Now we determine the analytical function $\mathbf{B}_1(\tau;\xi)$. By putting

$$\frac{d}{d\mu} = b(\xi)\frac{d}{d\xi}$$

we have

$$\mu = -\frac{1}{8\beta\gamma}\ln[2\chi\xi^2(2\chi\xi^2+\gamma)^{-1}] \tag{10}$$

$$\xi = (\gamma/2\chi)^{1/2}\exp(-4\beta\gamma\mu)\,[1-\exp(-8\beta\gamma\mu)]^{-1/2} \tag{11}$$

From equations (8) and (11), we obtain

$$\mathbf{B}_1(\tau;\mu) = \exp(\tau\, d/d\mu)\,[(\gamma/2\chi)^{1/2}\exp(-4\beta\gamma\mu)\,[1-\exp(-8\beta\gamma\mu)]^{-1/2}]$$

Finally, the function $\mathbf{B}_1(\tau;\mu)$ can be written in terms of ξ as

$$\mathbf{B}_1(\tau;\xi) = \gamma^{1/2}\xi\exp(-4\beta\gamma\tau)\,\{2\chi\xi^2[1-\exp(-8\beta\gamma\tau)]+\gamma\}^{-1/2} \tag{12}$$

where we have used

$$\exp(k\, d/d\xi)\, f(\xi) = f(\xi + k)$$

for an arbitrary function $f(\xi)$ and $dk/d\xi$.

Then we consider the series

$$\mathbf{B}_2(t;\xi) = \xi + \hat{\mathbf{I}}^2(t)b\{\xi + \hat{\mathbf{I}}^2(t)b\{\ldots\}\} \tag{13}$$

The function $\mathbf{B}_2(t;\xi)$ is related to the function $\mathbf{B}_1(\tau;\xi)$ by

$$\mathbf{B}_2(t;\xi) = \hat{f}^{(-)}(t;\eta)\hat{f}^{(+)}(\eta^2;\tau)\mathbf{B}_1(t;\xi) \tag{14}$$

where the operators $\hat{f}^{(+)}$ and $\hat{f}^{(-)}$ are defined as

$$\hat{f}^{(+)}(\eta;z)z^n = n!\,\eta^n, \qquad \hat{f}^{(-)}(z;\eta)\eta^n = \frac{1}{n!}z^n$$

Comparing equation (5) with equation (13) yields

$$\hat{\mathbf{q}}(t) = \exp\left[\hat{\mathbf{X}}(t)\frac{d}{d\xi}\right]\mathbf{B}_2(t;\xi)\Big|_{\xi=0} \tag{15}$$

Substituting equation (14) into equation (15) leads to

$$\hat{\mathbf{q}}(t) = \exp[\hat{\mathbf{X}}(t)\, d/d\xi]\,\hat{f}^{(-)}(t;\eta)\hat{f}^{(+)}(\eta^2,\tau)\mathbf{B}_1(\tau;\xi)\big|_{\xi=0} \tag{16}$$

Therefore the operator $\hat{\mathbf{q}}(t)$ is expressed in a compact form. It can be proved that

$$[\hat{\mathbf{X}}(t)]^n\hat{f}^{(-)}(t;\eta) = \hat{f}^{(-)}(t;\eta)[\hat{\mathbf{X}}(\eta)]^n$$

where

$$\hat{\mathbf{X}}(\eta) = \hat{\mathbf{q}}(0) + 2\beta\eta\,\hat{\mathbf{p}}(0) + \mathbf{K}(\eta)$$
$$\mathbf{K}(\eta) = \sum_{n=0}^{\infty} [-2\beta f^{n}(0)\eta + g^{n}(0)]\eta^{n+1} \tag{17}$$

Then equation (16) becomes

$$\hat{\mathbf{q}}(t) = \hat{j}^{(-)}(t;\eta)\hat{E}\hat{j}^{(+)}(\eta^2, \tau)\mathbf{B}_1(\tau;\xi)\big|_{\xi=0} \tag{18}$$

with

$$\hat{E} = \exp[\hat{\mathbf{X}}(\eta)\, d/d\xi] \tag{19}$$

Introduce the creation and annihilation operators $\hat{a}^+$ and $\hat{a}$, which obey the commutation relation $[\hat{a}, \hat{a}^+] = 1$ and are related to the displacement and momentum operators of the oscillator by

$$\hat{\mathbf{q}}(0) = \sigma(\hat{a}^+ + \hat{a}) = (\hbar^2\beta/4\gamma)^{1/4}(\hat{a}^+ + \hat{a})$$
$$\hat{\mathbf{p}}(0) = i\theta(\hat{a}^+ - \hat{a}) = i(\hbar^2\gamma/4\beta)^{1/4}(\hat{a}^+ - \hat{a})$$

Hence equation (17) reduces to

$$\hat{\mathbf{X}}(\eta) = V(\eta)\hat{a}^+ + V^*(\eta)\hat{a} + \mathbf{K}(\eta) \tag{20}$$

with

$$V(\eta) = (\sigma + i2\theta\beta\eta)$$

Substituting equation (20) into equation (19) yields

$$\hat{E} = \exp\{[V(\eta)\hat{a}^+ + V^*(\eta)\hat{a} + K(\eta)]\, d/d\xi\}$$

By using the Baker–Hausdorff theorem (Louisell, 1973), we have

$$\hat{E} = \exp[V(\eta)\hat{a}^+\, d/d\xi] \cdot \exp[V^*(\eta)\hat{a}\, d/d\xi]$$
$$\times \exp\{[\tfrac{1}{2}V(\eta)V^*(\eta)]\, d^2/d\xi^2\} \cdot \exp[K(\eta)\, d/d\xi] \tag{21}$$

Therefore equation (18) becomes

$$\hat{\mathbf{q}}(t) = \hat{j}^{(-)}(t;\eta)\exp[V(\eta)\hat{a}^+\, d/d\xi]\exp[V^*(\eta)\hat{a}\, d/d\xi]$$
$$\times \exp[\tfrac{1}{2}|V(\eta)|^2\, d^2/d\xi^2]$$
$$\times \exp[K(\eta)\, d/d\xi]\hat{j}^{(+)}(\eta^2;\tau)\mathbf{B}_1(\tau;\xi)\big|_{\xi=0} \tag{22}$$

The expectation value of the operator $\hat{\mathbf{q}}(t)$ in the coherent state is given by

$$\langle\hat{\mathbf{q}}(t)\rangle = \pi^{-1/2}\hat{j}^{(-)}(t;\eta)\hat{j}^{(+)}(\eta^2;\tau)$$
$$\times \int_{-\infty}^{\infty} d\bar{\xi}\exp(-\bar{\xi}^2)\,\mathbf{B}_1[\tau;\xi = \sqrt{2}\,|V(\eta)|\bar{\xi}$$
$$+ V(\eta)\alpha^* + V^*(\eta)\alpha + \mathbf{K}(\eta)] \tag{23}$$

where we have used

$$\exp(kd^2/d\xi^2)f(\xi) = (4\pi k)^{-1/2}\int_{-\infty}^{\infty} d\bar{\xi}\exp[-(\xi-\bar{\xi})^2/4k]\,f(\bar{\xi})$$

for an arbitrary function and $dk/d\xi = 0$. Equation (23) is just the desired expression for the time evolution of the displacement operator of the velocity-dependent anharmonic oscillator.

Similarly, the expectation values of the operator $\hat{\mathbf{p}}(t)$ in the coherent state can be written as

$$\begin{aligned}\langle\hat{\mathbf{p}}(t)\rangle &= \pi^{-1/2}(2\beta)^{-1}\hat{f}^{(-)}(t;\eta)\hat{f}(\eta^2;\tau)\\ &\times\int_{-\infty}^{\infty} d\bar{\xi}\exp(-\bar{\xi}^2)\,\eta^{-1}\{\mathbf{B}_1[\tau;\xi=\sqrt{2}\,|V(\eta)|\bar{\xi}\\ &+V^*(\eta)\alpha+V(\eta)\alpha^*+\mathbf{K}(\eta)]\\ &-\mathbf{B}_1[\tau;\xi=\sqrt{2}\,\sigma\bar{\xi}+(\alpha+\alpha^*)\sigma+\mathbf{K}(0)]\}\end{aligned} \tag{24}$$

The expectation values of the operators $\hat{\mathbf{q}}(t)$ and $\hat{\mathbf{p}}(t)$ in equations (23) and (24) can be represented as a Laplace transform and a subsequent inverse Laplace transform of suitable functionals, i.e.,

$$\begin{aligned}\langle\hat{\mathbf{q}}(t)\rangle &= \pi^{-1/2}\hat{\mathscr{L}}^{-1}(t;\eta^{-1})\hat{\mathscr{L}}(\eta^{-2};\tau)\eta^{-1}\gamma^{1/2}\\ &\times\int_{-\infty}^{\infty} d\bar{\xi}\exp(-\bar{\xi}^2)\exp(-4\beta\gamma\tau)\\ &\times\Xi(\bar{\xi};\eta)\{2\chi[\Xi(\bar{\xi};\eta)]^2[1-\exp(-8\beta\gamma\tau)]+\gamma\}^{-1/2}\end{aligned} \tag{25}$$

$$\begin{aligned}\langle\hat{\mathbf{p}}(t)\rangle &= \pi^{-1/2}(2\beta)^{-1}\hat{\mathscr{L}}^{-1}(t;\eta^{-1})\hat{\mathscr{L}}(\eta^{-2};\tau)\eta^{-2}\gamma^{1/2}\\ &\times\int_{-\infty}^{\infty} d\bar{\xi}\exp(-\bar{\xi}^2)\,[\Xi(\bar{\xi};\eta)\exp(-4\beta\gamma\tau)\,\{2\chi[\Xi(\bar{\xi};\eta)]^2\\ &\times[1-\exp(-8\beta\gamma\tau)]+\gamma\}^{-1/2}-\Xi_0(\bar{\xi})\exp(-4\beta\gamma\tau)\\ &\times\{2\chi[\Xi_0(\bar{\xi})]^2[1-\exp(-8\beta\gamma\tau)]+\gamma\}^{-1/2}]\end{aligned} \tag{26}$$

with

$$\Xi(\bar{\xi};\eta) = \sqrt{2}\,|V(\eta)|\bar{\xi}+V^*(\eta)\alpha+V(\eta)\alpha^*+\mathbf{K}(\eta)$$

$$\Xi_0(\bar{\xi}) = \sqrt{2}\,\sigma\bar{\xi}+\sigma(\alpha+\alpha^*)+\mathbf{K}(0)$$

$$\hat{f}^{(+)}(\eta;z) = \hat{\mathscr{L}}(\eta^{-1};z)\eta^{-1}, \qquad \hat{f}^{(-)}(z;\eta) = \hat{\mathscr{L}}^{(-)}(z;\eta^{-1})\eta$$

where $\hat{\mathscr{L}}(\eta;z)$ and $\hat{\mathscr{L}}^{-1}(z;\eta)$ are the Laplace and inverse Laplace transforms.

3. CONCLUSIONS

The Hamiltonian of the quantum forced anharmonic oscillator with a velocity-dependent term is considered, and the time development of the anharmonic oscillator is studied. The equation of motion for the displacement and momentum operators of the oscillator are solved by using the integral operators and iteration method. The solutions are presented as a Laplace transform and a subsequent inverse Laplace transform of suitable functionals of the creation and annihilation operators of the oscillator. These results can be used to analyze the properties of the velocity-dependent forced anharmonic oscillator and to obtain the physical quantities associated with the nonlinear processes in quantum optics without recourse to perturbation theory and variational calculus (Saavedra and Buendia, 1990). The functional integrals and their transforms in the above expressions can be evaluated by applying the convolution theorem and other effective methods.

REFERENCES

Carusotto, S. (1988). *Physical Review A*, **38**, 3249.

Louisell, W. H. (1973). *Quantum Statistical Properties of Radiation*, Wiley, New York, p. 137.

Saavedra, F. A., and Buendia, E. (1990). *Physical Review A*, **42**, 5073.

Zhang Jialun, Khan, R. D., Ding Sheng, and Shen Wenda (1992). *SPIE*, **1726**, to appear.